Indian Statistical Institute
Second Semester Examination 2004-2005
M.Math II Year
Fourier Analysis

Time: 3 hrs Date: 11-05-2005

Answer as many questions as you can. The maximum you can score is 105. Marks are indicated in the brackets. You may use your class-room notes in the exam. However you cannot use textbooks.

1. Find the Fourier series of the function defined by: $f(x) = 0$ if $\frac{\pi}{2} \leq |x| \leq \pi,\ f(x) = 0$ if x is rational and $|x| \leq \frac{\pi}{2},\ f(x) = 1$ if x is irrational and $|x| \leq \frac{\pi}{2}$, and repeated as a 2π periodic function. [10]

2. The Fourier coefficients of a 2π-periodic function f are given by:
$$\hat{f}(n) = ne^{-n^2}.$$
Prove that the Fourier series converges uniformly and absolutely to f. Find a *reasonable* value of N such that
$$\sup_{x \in I\!R} |S_N^f(x) - f(x)| \leq \frac{1}{100}.$$
(I don't want an absurd answer like $N = 10^5$!)
Justify your answer. [20]

3. Let f be a non-trivial L^1-function on $I\!R$. Let $x_N = \sum_{n=2}^{N} \frac{1}{n(\log n)^2}$. If $\hat{f}(x_N) = 0,\ N = 2, 3, 4 \ldots$ prove that f cannot be compactly supported. [15]

4. Let f be defined on the real line by: $f(x) = x^2$ if $|x| \leq 1,$ and $f(x) = 0$ otherwise. Compute the Fourier transform of f. [10]

5. Let $g_n,\ g \in \mathcal{S}(I\!R)$. We say $g_n \xrightarrow{s} g$, if for every $k \geq 0,\ l \geq 0$, $(1 + x^2)^l D^k g_n \to (1 + x^2)^l D^k g$ uniformly on $I\!R$. Decide if the following is true or not:
$$g_n \xrightarrow{s} g \Rightarrow (1 + x^2)^{25} g_n \xrightarrow{L^p} (1 + x^2)^{25} g, \quad \text{where } 1 \leq p < \infty.$$
If true give a proof; if false give a counter example. [15]

6. If $f \in \mathcal{S}(I\!R)$ and $g \in C_c^\infty(I\!R)$ and $f * g \equiv 0$, does it necessarily mean that $f \equiv 0$ or $g \equiv 0$? Justify your answer. [15]

7. Let $1 \leq p < \infty$ and f a C^2-function in $L^p(I\!R^n)$. If Δ is the Laplacian on $I\!R^n$, prove using facts about distributions of Fourier transforms that

$$\Delta f \equiv 0 \Rightarrow f \equiv 0.$$

(i.e. there are no non-trivial L^p harmonic functions if $1 \leq p < \infty$.)

[15]

8. Let $f(x) = e^{-|x|}$. Prove that f is an L^1-function and that

$$\overline{\mathrm{Sp}\{{}^x f : \quad x \in I\!R\}} = L^1(I\!R^n).$$

[15]